Lecture 17: Cellular homology

Cellular complex

Lemma

Let $\{(X_i, x_i)\}_{i\in I}$ be well-pointed spaces. Then

$$\tilde{\mathrm{H}}_n(\bigvee_{i\in I} X_i) = \bigoplus_{i\in I} \tilde{\mathrm{H}}_n(X_i).$$

Proof.

Let

$$Y = \coprod_{i\in I} X_i, \quad A = \coprod_{i\in I} \{x_i\}$$

$A \subset Y$ is a cofibration, therefore

$$\tilde{\mathrm{H}}_n(\bigvee_{i\in I} X_i) = \tilde{\mathrm{H}}_n(Y/A) = \mathrm{H}_n(Y, A) = \bigoplus_{i\in I} \mathrm{H}_n(X_i, x_i) = \bigoplus_{i\in I} \tilde{\mathrm{H}}_n(X_i).$$

□

Definition

Let (X, A) be a relative CW complex with skeletons:

$$A = X^{-1} \subset X^0 \subset \cdots \subset X^n \subset \cdots .$$

We define the relative cellular chain complex $(C_\bullet^{cell}(X, A), \partial)$

$$\cdots \to C_n^{cell}(X, A) \xrightarrow{\partial} C_{n-1}^{cell}(X, A) \xrightarrow{\partial} \cdots \to C_0^{cell}(X, A) \to 0$$

where

$$C_n^{cell}(X, A) := \mathrm{H}_n(X^n, X^{n-1})$$

and the boundary map ∂ is defined by the commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(X^n, X^{n-1}) & \xrightarrow{\quad\quad\partial\quad\quad} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \\ & {\scriptstyle\delta}\searrow \qquad \nearrow{\scriptstyle j} & \\ & \mathrm{H}_{n-1}(X^{n-1}, A) & \end{array}$$

Assume X^n is obtained from X^{n-1} by attaching n-cells

$$\begin{array}{ccc} \coprod_{\alpha\in J_n} S^{n-1} & \xrightarrow{f} & X^{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{\alpha\in J_n} D^n & \xrightarrow{\Phi_f} & X^n \end{array}$$

Since $X^{n-1} \hookrightarrow X^n$ is a cofibration,

$$C_n^{cell}(X,A) \simeq \tilde{\mathrm{H}}_n(X^n/X^{n-1}) \simeq \bigoplus_{J_n} \tilde{\mathrm{H}}_n(S^n) \simeq \bigoplus_{J_n} \mathbb{Z}$$

is the free abelian group generated by each attached

$$\mathrm{H}_n(D^n, S^{n-1}) = \tilde{\mathrm{H}}_n(S^n).$$

Using the diagram

$$
\begin{array}{ccccc}
\mathrm{H}_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{\partial_{n-1}} & \mathrm{H}_{n-2}(X^{n-2}, X^{n-3}) \\
\downarrow{\scriptstyle \delta_n} & \nearrow{\scriptstyle j_n} & & \searrow{\scriptstyle \delta_{n-1}} & \uparrow{\scriptstyle j_{n-1}} \\
\mathrm{H}_{n-1}(X^{n-1}, A) & & & & \mathrm{H}_{n-1}(X^{n-2}, A)
\end{array}
$$

and $\delta_{n-1} \circ j_n = 0$, we see that

$$\partial_{n-1} \circ \partial_n = j_{n-1} \circ \delta_{n-1} \circ j_n \circ \delta_n = 0.$$

Therefore $(C_\bullet^{cell}(X, A), \partial)$ indeed defines a chain complex.

Definition

Let (X, A) be a relative CW complex. We define its n-th relative cellular homology by

$$\boxed{\mathrm{H}_n^{cell}(X, A) := \mathrm{H}_n(C_\bullet^{cell}(X, A), \partial)}.$$

When $A = \emptyset$, we simply denote it by $\mathrm{H}_n^{cell}(X)$ called the n-th cellular homology.

Lemma

Let (X, A) be a relative CW complex. Let $0 \leq q < p \leq \infty$. Then

$$\mathrm{H}_n(X^p, X^q) = 0, \quad \text{if} \quad n \leq q \quad \text{or} \quad n > p.$$

Proof: Consider the cofibrations

$$X^q \hookrightarrow X^{q+1} \hookrightarrow \cdots \hookrightarrow X^{p-1} \hookrightarrow X^p$$

where each quotient is a wedge of spheres

$$X^{q+1}/X^q = \bigvee S^{q+1}, \quad \cdots, \quad X^p/X^{p-1} = \bigvee S^p.$$

Assume $n \leq q$ or $n > p$. Then

$$\mathrm{H}_n(X^{q+1}, X^q) = \mathrm{H}_n(X^{q+2}, X^{q+1}) = \cdots = \mathrm{H}_n(X^p, X^{p-1}) = 0.$$

Consider the triple $X^q \hookrightarrow X^{q+1} \hookrightarrow X^{q+2}$. The exact sequence

$$\mathrm{H}_n(X^{q+1}, X^q) \to \mathrm{H}_n(X^{q+2}, X^q) \to \mathrm{H}_n(X^{q+2}, X^{q+1})$$

implies $\mathrm{H}_n(X^{q+2}, X^q) = 0$.

The same argument applying to the triple $X^q \hookrightarrow X^{q+2} \hookrightarrow X^{q+3}$ implies $\mathrm{H}_n(X^{q+3}, X^q) = 0$. Repeating this process until arriving at $X^q \hookrightarrow X^{p-1} \hookrightarrow X^p$, we find $\mathrm{H}_n(X^p, X^q) = 0$. $\square$

Theorem

Let (X, A) be a relative CW complex. Then the cellular homology coincides with the singular homology

$$\mathrm{H}_n^{cell}(X, A) \simeq \mathrm{H}_n(X, A).$$

Proof

Consider the following commutative diagram

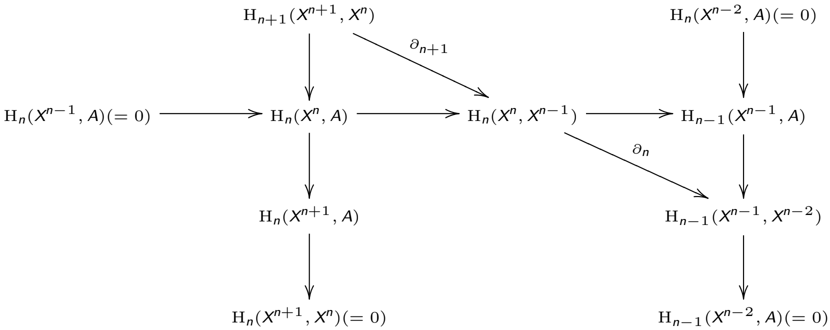


Diagram chasing implies

$$H_n(X^{n+1}, A) \simeq H_n^{cell}(X, A).$$

Theorem follows from the exact sequence

$$H_{n+1}(X, X^{n+1})(= 0) \to H_n(X^{n+1}, A) \to H_n(X, A) \to H_n(X, X^{n+1})(= 0)$$

Let $f\colon (X, A) \to (Y, B)$ be a cellular map. It induces a map on cellular homology

$$f_* : \mathrm{H}^{cell}_\bullet(X, A) \to \mathrm{H}^{cell}_\bullet(Y, B).$$

Therefore in the category of CW complexes, we can work entirely with cellular homology which is combinatorially easier to compute.

Cellular Boundary Formula

Let us now analyze cellular differential

$$\partial_n : \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}).$$

For each n-cell e^n_α, we have the gluing map

$$f_{e^n_\alpha} : S^{n-1} \to X^{n-1}.$$

This defines a map

$$\bar{f}_{e^n_\alpha} : S^{n-1} \to X^{n-1}/X^{n-2} = \bigvee_{J_{n-1}} S^{n-1}$$

which induces a degree map

$$(\bar{f}_{e^n_\alpha})_* : \tilde{\mathrm{H}}_{n-1}(S^{n-1}) \simeq \mathbb{Z} \to \bigoplus_{J_{n-1}} \tilde{\mathrm{H}}_{n-1}(S^{n-1}) \simeq \bigoplus_{J_{n-1}} \mathbb{Z}.$$

Collecting all n-cells, this generates the degree map

$$d_n : \bigoplus_{J_n} \mathbb{Z} \to \bigoplus_{J_{n-1}} \mathbb{Z}.$$

Theorem

Under the identification $C_n^{cell}(X^n, X^{n-1}) \simeq \bigoplus_{J_n} \mathbb{Z}$, cellular differential coincides with the degree map

$$\partial_n \simeq d_n.$$

Proof.

This follows from chasing the definition of the connecting map $\partial_n : \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$.

□

Example

$\mathbb{CP}^n$ has a CW structure with a single $2m$-cell for each $m \leq n$. Since there is no odd dim cells, the degree map $d = 0$. We find

$$\mathrm{H}_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Example

A closed oriented surface Σ_g of genus g has a CW structure with a 0-cell e_0, 2g 1-cells $\{a_1, b_1, \cdots, a_g, b_g\}$, and a 2-cell e_2.
In the cell complex

$$\mathbb{Z}e_2 \xrightarrow{d_2} \bigoplus_i \mathbb{Z}a_i \oplus \bigoplus_i \mathbb{Z}b_i \xrightarrow{d_1} \mathbb{Z}e_0.$$

the degree map d_2 sends

$$e_2 \to \sum_i (a_i + b_i - a_i - b_i) = 0$$

so $d_2 = 0$. Similarly, d_1 is also 0. We find

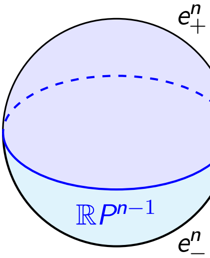
$$\mathrm{H}_k(\Sigma_g) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{2g} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k > 2. \end{cases}$$

Example

$\mathbb{R}P^n = S^n/\mathbb{Z}_2$ has a CW structure with a k-cell for each $0 \le k \le n$.

$$\mathbb{R}P^0 \hookrightarrow \mathbb{R}P^1 \hookrightarrow \cdots \hookrightarrow \mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n.$$

$\mathbb{R}P^n$



e^n_+ and e^n_- are identified under $\mathbb{Z}_2$

We have the cell complex

$$\mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \to \cdots \xrightarrow{d_1} \mathbb{Z}$$

The degree map $d_k : \tilde{\mathrm{H}}_{k-1}(S^{k-1}) \to \mathrm{H}_{k-1}(S^{k-1})$ is

$$d_k = 1 + \deg(\text{antipodal map}) = 1 + (-1)^k.$$

So the cell complex becomes

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

It follows that

$$\mathrm{H}_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n, k \text{ odd} \\ \mathbb{Z} & k = n = \text{odd} \\ 0 & k = n = \text{even} \\ 0 & k > n \end{cases}$$

Euler characteristic

Definition

Let X be a finite CW complex of dimension n and denote by c_i the number of i-cells of X. The Euler characteristic of X is defined as:

$$\chi(X) := \sum_i (-1)^i c_i.$$

Recall that any finitely generaed abelian group G is decomposed into a free part and a torsion part

$$G \simeq \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z}.$$

The integer $r := \mathrm{rk}(G)$ is called the rank of G.

Lemma

Let $(G_\bullet, \partial)$ be a chain complex of finitely generaed abelian groups such that $G_n = 0$ if $|n| >> 0$. Then

$$\sum_i (-1)^i \mathrm{rk}(G_i) = \sum_i (-1)^i \mathrm{rk}(\mathrm{H}_i(G_\bullet)).$$

Proof.

We consider the chain complex $(G_\bullet^{\mathbb{Q}}, \partial)$ where

$$G_k^{\mathbb{Q}} = G_k \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^{\mathrm{rk}(G_k)}.$$

Each $G_k^{\mathbb{Q}}$ is now a vector space over the field $\mathbb{Q}$, and ∂ is a linear map. Moreover

$$\mathrm{H}_i(G_\bullet^{\mathbb{Q}}) = \mathbb{Q}^{\mathrm{rk}(\mathrm{H}_i(G_\bullet))}.$$

The lemma follows from the corresponding statement for linear maps on vector spaces. □

Theorem

Let X be a finite CW complex. Then

$$\chi(X) = \sum_i (-1)^i b_i(X)$$

where $b_i(X) := rk(\mathrm{H}_i(X))$ is called the i-th Betti number of X In particular, $\chi(X)$ is independent of the chosen CW structure on X and only depend on the cellular homotopy class of X.

Example

$\chi(S^n) = 1 + (-1)^n$.

Example

Let X be the tetrahedron and Y be the cube. They give two different CW structures on S^2

They give two different counts of the Euler characteristic of S^2

$$\chi(X) = 4 - 6 + 4 = 2, \quad \chi(Y) = 8 - 12 + 6 = 2.$$